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Some Results on Differential Subordinations for a class of functions defined using Generalized Ruscheweyh Derivative operator

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Abstract

By making use of the principle of subordination, we introduce a subclass of analytic functions involving generalized Ruscheweyh Derivative operator. The properties such as inclusion relationships, distortion theorems, coefficient inequalities for the above class have been discussed. For analytic functions defined in open unit disc, we establish the differential sandwich theorem.

1 Introduction

Let A_p denote the class of functions that are analytic in the unit disk $\Delta = \{z : z \in C, |z| < 1\}$ and consisting of the functions f of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}, (p, n \in N = \{1, 2, 3...\}) \tag{1.1}$$

where f is analytic and p-valent in Δ. If $f \in A_p$ is given by (1.1) and $g \in A_p$ is given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}, (p, n \in N = \{1, 2, 3...\}) \tag{1.2}$$

then the Hadamard product(or convolution) $f * g$ of f and g, defined by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z). \tag{1.3}$$

Let f and g are analytic functions defined in Δ. The function f is said to be *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in Δ with $w(z) = 0, |w(z)| < 1, z \in \Delta$ such that

$$f(z) = g(w(z)), \quad (z \in \Delta)$$

We denote this subordination by $f \prec g$ or $f(z) \prec g(z), \quad (z \in \Delta)$.

In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, $(z \in \Delta)$.

Let $H(\Delta)$ denote a class of analytic functions in Δ and let $H(a, n)$ denote a subclass of functions $f \in H(\Delta)$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

By Q we denote the set of all functions $f(z)$ analytic and injective on $\Delta \setminus E(f)$ (see [8]), where

$$E(f) = \{\xi \in \partial\Delta : \lim_{z \to \xi} f(z) = \infty\},$$

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and such that $f'(\xi) \neq 0$ for $\xi \in \partial\Delta \setminus E(f)$ where $\partial\Delta =$ boundary of Δ.

Let $\psi : C^3 \times \Delta \to C$, let $h(z)$ be univalent in Δ, and let $q(z) \in Q$. Miller and Mocanu [7] considered the problem of finding the conditions imposed on an admissible function ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.4}$$

implies that $p(z) \prec q(z)$ for all functions $p(z) \in H(a,n)$ satisfying the differential subordination (1.4). Moreover, they established conditions under which $q(z)$ is the smallest function with this property. It is called the best dominant of subordination (1.4).

Let $\varphi : C^3 \times \Delta \to C$, let $h(z) \in H$, and let $q(z) \in H(a,n)$. Recently, Miller and Mocanu [8] studied the dual problem and established the conditions for φ under which

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.5}$$

implies that $q(z) \prec p(z)$ for all functions $q(z) \in Q$ satisfying the superordination considered above. They also found conditions under which $q(z)$ is the largest function with this property. It is called the best subordinant of superordination (1.5).

Definition 1.1 *Let f(z) is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Then the* generalized fractional derivative of order λ *is defined for a function f(z) by (see, e.g. [**?**])*

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \left\{ z^{\lambda-\mu} \displaystyle\int_0^z (z-\zeta)^{-\lambda} \cdot {}_2F_1\left(\mu-\lambda, -\nu; 1-\lambda; 1-\dfrac{\zeta}{z}\right) f(\zeta) d\zeta \right\}, \\ (0 \leq \lambda < 1) \\ \dfrac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), (n \leq \lambda < n+1, n \in N) \end{cases} \tag{1.6}$$

and $f(z) = O(|z|^k), (z \to 0, k > max\{0, \mu-\nu-1\} - 1)$

It follows at once from the above definition that

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) := D_z^\lambda f(z), (0 \leq \lambda < 1)$$

Furthermore, in terms of gamma function, we have:

$$J_{0,z}^{\lambda,\mu,\nu} z^\rho := \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+1)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu}, \tag{1.7}$$

$(0 \leq \lambda;\ \rho > max\{0, \mu-\nu-1\} - 1)$

Definition 1.2 Generalized Ruscheweyh derivative *introduced by Goyal and Goyal [2](see also [1], [9]) is defined as*

$$\mathbb{J}_p^{\lambda,\mu} f(z) := \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\mu+1)\Gamma(\nu+2)} z^p J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-p} f(z)) \tag{1.8}$$

$$= z^p + \sum_{k=n+p}^{\infty} a^k B_p^{\lambda,\mu}(k) z^k$$

where

$$B_p^{\lambda,\mu}(k) := \frac{\Gamma(k+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu+1)}{\Gamma(k)\Gamma(k+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)},\ (\lambda, \mu, \nu \in R \text{ and } 0 \leq \lambda < 1)$$

For $\mu = \lambda$, generalized Ruscheweyh derivative $\mathbb{J}_p^{\lambda,\mu}$ reduces to the ordinary Ruscheweyh derivative D^λ of order λ.

Equation (1.8) can also be written in convolution form as:

$$\mathbb{J}_{0,z}^{\lambda,\mu,\nu} f(z) =_2 F_1(\mu+1, \nu+2; \nu+2+\mu-\lambda; z) * f(z)$$

Motivated by [13], and by making use of the operator $\mathbb{J}_p^{\lambda+1,\mu}$ and principle of subordination, we introduce and investigate a new subclass of class $A_n(p)$ of p-valent analytic functions as follows:

Definition 1.3 *A function $f \in A_p(n)$ is said to be in class $R_p^{\lambda,\mu}(\alpha;\phi)$ if it satisfies the following subordination condition:*

$$(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \prec \phi(z) \tag{1.9}$$

where

$$(\lambda,\mu,\nu \in R \ \text{ and } \ 0 \le \lambda < 1)$$

Special Cases:

1. *If $\phi(z) = \frac{1+Az}{1+Bz}$, we denote the class $R_p^{\lambda,\mu}(\alpha;A,B)$.*
2. *If $\phi(z) = q(z) + \alpha z q'(z)$, we denote the class $R_p^{\lambda,\mu}(\alpha;q)$.*
3. *If $\lambda = \mu$, then $R_p^{\lambda,\lambda}(\alpha;\phi) = R_p^{\lambda}(\alpha;\phi)$*
4. *If $\lambda = \mu$ and $p = 1$, then $R_1^{\lambda,\lambda}(\alpha;\phi) = R^{\lambda}(\alpha;\phi)$*

2 Preliminaries

In order to prove our main results, we need the following lemmas:

Lemma 2.1 *[7] Let the function hbe analytic and convex(univalent) in Δ with $h(0) = 1$. Suppose also that the function k given by*

$$k(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + ...$$

is analytic in Δ. If

$$k(z) + \frac{zk'(z)}{\zeta} \prec h(z) \quad (R(\zeta) > 0; \zeta \ne 0; z \in \Delta) \tag{2.1}$$

then

$$k(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1} h(t)dt \prec h(z) \quad (z \in \Delta)$$

and $\chi(z)$ is the best dominant of (2.1).

Lemma 2.2 *[4] Let w be a non-constant analytic function in Δ with $w(0) = 0$. If $|w|$ attains it maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = k\, w(z_0),$$

where $k \ge 1$ is a real number.

Lemma 2.3 *[5] Let F be analytic and convex in Δ. If $f, g \in A$ and $f, g \prec F$, then $\lambda f + (1-\lambda)g \prec F \quad (0 \le \lambda \le 1)$.*

Lemma 2.4 *[12] Let $q(z)$ be univalent in Δ, let $\gamma \in C^* = C \setminus \{0\}$,and let*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)} > \max\left\{0, -Re\frac{1}{\gamma}\right\}\right\}.$$

If $p(z)$ is analytic in Δ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.5 *[8] Let $q(z)$ is convex in Δ, let $q(0) = a$, let $\gamma \in C, Re\gamma > 0$. If $p \in H(a,1)$ and $p(z) + \gamma z p'(z)$ is univalent in Δ, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z)$$

where $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Lemma 2.6 *[11] Let $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$ be analytic in Δ and $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ be analytic and convex in Δ. If $f \prec g$, then $|a_k| \le |b_1| \quad (k \in N)$.*

3 Main Results

Theorem 3.1 *Recurrence relation for generalized Ruscheweyh derivative operator defined in (1.8) is given by*

$$z\left[\mathbb{J}_p^{\lambda,\mu}f(z)\right]' = (\mu-\lambda+\nu+1)\mathbb{J}_p^{\lambda+1,\mu}f(z) - (\mu-\lambda+\nu+1-p)\mathbb{J}_p^{\lambda,\mu}f(z) \tag{3.1}$$

Proof. Using equation (1.8) and (1.7), we get the above result.

Theorem 3.2 *Let $f \in R_p^{\lambda,\mu}(\alpha;\phi)$ with $\alpha \in C$, $Re(\alpha) > 0$, then*

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \prec \frac{1}{n\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{1}{n\alpha}-1}du \prec \frac{1+Az}{1+Bz}, \quad (z \in \Delta) \tag{3.2}$$

Proof. Define the function p_1 by

$$p_1(z) = \frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}, \quad (z \in \Delta) \tag{3.3}$$

then p_1 is analytic in Δ with $p_1(0) = 1$. By taking the derivative both side in equality (3.3) and using (3.1), we get

$$\begin{aligned} &(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \\ &= p_1(z) + \alpha z p_1'(z) \prec \frac{1+Az}{1+Bz}, \quad (z \in \Delta) \end{aligned} \tag{3.4}$$

An application of Lemma 2.1 to (3.4) yields

$$\begin{aligned} \frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} &\prec \frac{1}{n\alpha}\int_0^z \frac{1+At}{1+Bt}t^{\frac{1}{n\alpha}-1}dt \\ &= \frac{1}{n\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{1}{n\alpha}-1}du \prec \frac{1+Az}{1+Bz}, \quad (z \in \Delta) \end{aligned}$$

The proof of Theorem 3.2 is thus completed.

Theorem 3.3 *Let $q(z)$ be univalent in Δ with $q(0) = 1$ and $\alpha \in C, Re(\alpha) > 0$. suppose that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, \ -Re\frac{1}{\alpha}\right\} \tag{3.5}$$

If $f(z) \in A_p(n)$ satisfies the following subordination

$$\begin{aligned} &(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \\ &\prec q(z) + \alpha z q'(z) \end{aligned} \tag{3.6}$$

then

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \prec q(z)$$

and $q(z)$ is best dominant.

Proof. Let

$$p(z) = \frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \tag{3.7}$$

differentiating (3.7) w.r.to z and using the identity (3.1) in the following resulting equation, we find

$$zp'(z) = (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - (\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 + \frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}$$

Hence we get

$$p(z)+\alpha zp'(z) = (1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]$$

By the relation (3.6), we obtain

$$p(z)+\alpha zp'(z) \prec q(z)+\alpha zq'(z)$$

According to Lemma 2.4

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \prec q(z)$$

The proof of Theorem 3.3 is thus completed.

Taking a convex function $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4 *Let $\alpha \in C$, $Re(\alpha) > 0$ and $-1 \le A < B \le 1$. If $f(z) \in A_p(n)$ satisfies the following subordination*

$$R_p^{\lambda,\mu}(\alpha;\phi) \prec \frac{1+Az}{1+Bz} + \alpha\frac{(A-B)z}{(1+Bz)^2}$$

where $R_p^{\lambda,\mu}(\alpha;\phi)$ is given by (1.9), then

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3.5 *Let $q(z)$ be convex in Δ, let $q(0) = 1$ and let $\alpha \in C$, $Re(\alpha) > 0$. If $f(z) \in A_p(n)$ is such that*

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \in H(q(0),1)\cap Q$$

and $R_p^{\lambda,\mu}(\alpha;\phi)$ is univalent in Δ and satisfies the superordination

$$q(z)+\alpha zq'(z) \prec R_p^{\lambda,\mu}(\alpha;\phi) \tag{3.8}$$

where $R_p^{\lambda,\mu}(\alpha;\phi)$ is given by (1.9), then

$$q(z) \prec \frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}$$

and $q(z)$ is the best subordinant.

Proof. Let $p(z)$ be given by (3.7). Proceeding as in the proof of Theorem 3.3, we rewrite subordination (3.8) in the form

$$q(z)+\alpha zq'(z) \prec p(z)+\alpha zp'(z)$$

The proof is obtained by applying Lemma 2.5.

Corollary 3.6 *Let $\alpha \in C$, $Re(\alpha) > 0$ and $-1 \le A < B \le 1$. If $f(z) \in A_p(n)$ is such that*

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \in H(q(0),1)\cap Q$$

and $R_p^{\lambda,\mu}(\alpha;\phi)$ is univalent in Δ and satisfies the superordination

$$\frac{1+Az}{1+Bz}+\alpha\frac{(A-B)z}{(1+Bz)^2}\prec R_p^{\lambda,\mu}(\alpha;\phi)$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Combining Theorem 3.3 and Theorem 3.5, we arrive at the following Sandwich theorem:

Theorem 3.7 *Let $q_1(z)$ and $q_2(z)$ be convex in Δ, let $q_1(0)=q_2(0)=1$, let $q_2(z)$ satisfy (3.5) and let $\alpha\in C$, $Re(\alpha)>0$. If $f(z)\in A_p(n)$ is such that*

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\in H(q(0),1)\cap Q$$

and $R_p^{\lambda,\mu}(\alpha;\phi)$ is univalent in Δ satisfies the relation

$$q_1(z)+\alpha zq_1'(z)\prec R_p^{\lambda,\mu}(\alpha;\phi)\prec q_2(z)+\alpha zq_2'(z)$$

where $R_p^{\lambda,\mu}(\alpha;\phi)$ is given by(1.9), then

$$q_1(z)\prec\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are the best subordinant and best dominant respectively.

Remark: Combining Corollaries 3.4 and 3.6, we obtain the corresponding Sandwich results for the operators $\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}$

Theorem 3.8 *Let $f\in R_p^{\lambda,\mu}(\alpha;A,B)$ with $\alpha>0$, $Re(\alpha)>0$ and $-1\leq B<A\leq 1$, then*

$$\frac{1}{n\alpha}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{1}{n\alpha}-1}du<Re\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)<\frac{1}{n\alpha}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{1}{n\alpha}-1}du \tag{3.9}$$

Proof. Let $f\in R_p^{\lambda,\mu}(\alpha;A,B)$ with $Re(\alpha)>0$. From Theorem 3.2, we know that equation (3.2) holds, which implies that

$$\begin{aligned}Re\left[\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]&<\sup_{z\in\Delta}Re\left[\frac{1}{n\alpha}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{1}{n\alpha}-1}du\right]\\&\leq\frac{1}{n\alpha}\int_0^1 sup_{z\in\Delta}Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{1}{n\alpha}-1}du\\&<\frac{1}{n\alpha}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{1}{n\alpha}-1}du\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}Re\left[\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]&>\inf_{z\in\Delta}Re\left[\frac{1}{n\alpha}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{1}{n\alpha}-1}du\right]\\&\geq\frac{1}{n\alpha}\int_0^1 inf_{z\in\Delta}Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{1}{n\alpha}-1}du\\&>\frac{1}{n\alpha}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{1}{n\alpha}-1}du\end{aligned}\tag{3.11}$$

Combining equation (3.10) and (3.11), we get

$$\frac{1}{n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{1}{n\alpha}-1}du < Re\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right) < \frac{1}{n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{1}{n\alpha}-1}du \tag{3.12}$$

Corollary 3.9 *Let $f \in R_p^{\lambda,\mu}(\alpha; A, B)$ with $\alpha > 0$, $Re(\alpha) > 0$ and $-1 \leq A < B \leq 1$, then*

$$\frac{1}{n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{1}{n\alpha}-1}du < Re\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right) < \frac{1}{n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{1}{n\alpha}-1}du \tag{3.13}$$

Proof. The result may be proved similar to the method of Theorem 3.8

Theorem 3.10 *Let $f(z) \in A_p(n)$, $\xi \in C \setminus \{0\}$ and $0 \leq \gamma \leq 1$ also let the function φ defined by*

$$\varphi(z) = \frac{\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]'}{\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]}, \quad (z \in \Delta)$$

If φ satisfies one of the following conditions:

$$\mathfrak{Re}(\varphi(z)) \begin{cases} < \frac{1}{|\xi|^2}\mathfrak{Re}(\xi) & (\mathfrak{Re}(\xi) > 0), \\ \neq 0 & (\mathfrak{Re}(\xi) = 0), \\ > \frac{1}{|\xi|^2}\mathfrak{Re}(\xi) & (\mathfrak{Re}(\xi) < 0), \end{cases} \tag{3.14}$$

or

$$\mathfrak{Im}(\varphi(z)) \begin{cases} > -\frac{1}{|\xi|^2}\mathfrak{Im}(\xi) & (\mathfrak{Im}(\xi) > 0), \\ \neq 0 & (\mathfrak{Im}(\xi) = 0), \\ < -\frac{1}{|\xi|^2}\mathfrak{Im}(\xi) & (\mathfrak{Im}(\xi) < 0), \end{cases} \tag{3.15}$$

then

$$\left|\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]^\xi\right| < 1-\gamma$$

Proof. We define the following function ϕ by

$$\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]^\xi = (1-\gamma)\phi(z) \tag{3.16}$$

$$(0 \leq \gamma < 1, \quad \xi \in C \setminus \{0\}, \quad z \in \Delta)$$

It is easy to say that the function $\phi(z)$ is analytic in Δ with $\phi(0) = 0$.

Differentiating both side of (3.16) w.r. to z logarithmically, we get

$$z\frac{\phi'(z)}{\phi(z)} = \xi\frac{z\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]'}{\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]} \quad (z \in \Delta; \quad \xi \in C \setminus \{0\})$$

Now we consider the function φ defined by

$$\varphi = \frac{\bar{\xi}}{|\xi|^2}\frac{z\phi'(z)}{\phi(z)} \quad (z \in \Delta;\quad \xi \in C \setminus \{0\}) \tag{3.17}$$

Assume that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z|\leq|z_0|} |\phi(z)| = |\phi(z_0)| = 1$$

by Lemma 2.2, we know that

$$z\phi'(z_0) = k\phi(z_0) \quad (k \geq 1) \tag{3.18}$$

It follows from (3.17) and (3.18), that

$$\mathfrak{Re}(\varphi(z_0)) = \mathfrak{Re}\left(\frac{\bar{\xi}}{|\xi|^2}\frac{z_0\phi'(z_0)}{\phi(z_0)}\right) = \mathfrak{Re}\left(\frac{\bar{\xi}}{|\xi|^2}\ k\right) = \frac{k}{|\xi|^2}\mathfrak{Re}(\xi)\begin{cases} \geq \dfrac{1}{|\xi|^2}\mathfrak{Re}(\xi) & (\mathfrak{Re}(\xi) > 0), \\ = 0 & (\mathfrak{Re}(\xi) = 0), \\ \leq \dfrac{1}{|\xi|^2}\mathfrak{Re}(\xi) & (\mathfrak{Re}(\xi) < 0), \end{cases} \tag{3.19}$$

and

$$\mathfrak{Im}(\varphi(z_0)) = \mathfrak{Im}\left(\frac{\bar{\xi}}{|\xi|^2}\frac{z_0\phi'(z_0)}{\phi(z_0)}\right) = \mathfrak{Im}\left(\frac{\bar{\xi}}{|\xi|^2}\ k\right) = -\frac{k}{|\xi|^2}\mathfrak{Im}(\xi)\begin{cases} \leq -\dfrac{1}{|\xi|^2}\mathfrak{Im}(\xi) & (\mathfrak{Im}(\xi) > 0), \\ = 0 & (\mathfrak{Im}(\xi) = 0), \\ \geq -\dfrac{1}{|\xi|^2}\mathfrak{Im}(\xi) & (\mathfrak{Im}(\xi) < 0), \end{cases} \tag{3.20}$$

But the inequalities in (3.19) and (3.20) contradict respectively the inequalities in (3.14) and (3.15).

Therefore we can conclude that $|\phi(z)| < 1, \quad (z \in \Delta)$
which implies that

$$\left|\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - 1\right]^{\xi}\right| = (1-\gamma)|\phi(z)| < 1-\gamma$$

This completes the proof.

Theorem 3.11 *Let* $Re(\alpha) > 0,$ *and* $f \in R_p^{\lambda,\mu}(0; 1-2\rho, -1)$ $(0 \leq \rho < 1)$ *then* $f \in R_p^{\lambda,\mu}(\alpha; 1-2\rho, -1)$ *for* $|z| < K(\alpha)$, *where*

$$K(\alpha) = -\alpha + \sqrt{(\alpha^2+1)} \tag{3.21}$$

The bound $K(\alpha)$ *is the best possible.*

Proof. Suppose that

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} = \rho + (1-\rho)h(z), \quad (z \in \Delta; 0 \leq \rho < 1)$$

Where h is analytic and has a positive real part in Δ. By taking the derivative in the both side and using recurrence relation (3.1), we get

$$(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]$$
$$= p(z) + \alpha z p'(z)$$

ie

$$(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}-\alpha\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2-(\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]$$
$$=\rho+(1-\rho)h(z)+\alpha z(1-\rho)h'(z)$$

$$\begin{aligned}&\mathfrak{Re}\left[(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}-\alpha\left\{(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2-(\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right\}-\rho\right]\\&=(1-\rho)\,\mathfrak{Re}(h(z)+\alpha zh'(z))\\&\geq(1-\rho)\,\mathfrak{Re}(h(z)-\alpha|zh'(z)|)\end{aligned}\tag{3.22}$$

By making the use of the following well-known estimate (see [6]):

$$|zh'(z)|\leq\frac{2r}{1-r^2}\mathfrak{Re}(h(z)),\quad (|z|=r<1)$$

in (3.22), we obtain that

$$\begin{aligned}&\mathfrak{Re}\left[(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}-\alpha\left\{(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2-(\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right\}-\rho\right]\\&\geq(1-\rho)\left(1-\frac{2\alpha r}{1-r^2}\right)\mathfrak{Re}(h(z))>0\end{aligned}$$

for $r<K(\alpha)$ where $K(\alpha)$ is given by (3.21).

To show that the bound $K(\alpha)$ is the best possible, we consider the function $f\in A_p(n)$ defined by

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}=\rho-(1-\rho)\left(\frac{1-z}{1+z}\right),\quad (z\in\Delta)$$

By noting that

$$\begin{aligned}&\mathfrak{Re}\left[(1+\alpha)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}-\alpha\left\{(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2-(\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right\}-\rho\right]\\&=(1-\rho)\,\mathfrak{Re}\left(\frac{1-z}{1+z}-\frac{2\alpha z}{(1+z)^2}\right)=0\end{aligned}$$

for $z=K(\alpha)$, we conclude that the bound is the best possible. Theorem 3.11 is proved.

Theorem 3.12 *Let $Re(\alpha_2)\geq Re(\alpha_1)\geq 0$ and $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$. Then*

$$R_p^{\lambda,\mu}(\alpha_2;A_2,B_2)\subset R_p^{\lambda,\mu}(\alpha_1;A_1,B_1)\tag{3.23}$$

Proof. Suppose that $f\in R_p^{\lambda,\mu}(\alpha_2;A_2,B_2)$, we know that

$$(1+\alpha_2)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}-\alpha_2\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2-(\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right]\prec\frac{1+A_2z}{1+B_2z}$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$ we easily find that

$$(1+\alpha_2)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha_2\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z} \tag{3.24}$$

that is $f \in R_p^{\lambda,\mu}(\alpha_1; A_1, B_1)$

Thus assertion of Theorem 3.12 holds for $\alpha_2 = \alpha_1 \geq 0$

If $\alpha_2 > \alpha_1 \geq 0$, by Theorem 3.2 and (3.24) we know that $f \in R_p^{\lambda,\mu}(0; A_1, B_1)$ that is

$$\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \prec \frac{1+A_1z}{1+B_1z} \tag{3.25}$$

At the same time, we have

$$\begin{aligned}(1+\alpha_1)&\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha_1\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \\ &= \left(1-\frac{\alpha_1}{\alpha_2}\right)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} \\ &+ \frac{\alpha_1}{\alpha_2}\left[(1+\alpha_2)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha_2\left\{(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right\}\right]\end{aligned} \tag{3.26}$$

Moreover since $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and $h_1(z) = \frac{1+A_1z}{1+B_1z}$, $(z \in \Delta)$ is analytic and convex in Δ. Combining (3.24), 3.25 and (3.26) and Lemma 2.3, we find that

$$(1+\alpha_1)\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)} - \alpha_1\left[(\mu-\lambda+\nu+1)\left(\frac{\mathbb{J}_p^{\lambda+1,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right)^2 - (\mu-\lambda+\nu)\frac{\mathbb{J}_p^{\lambda+2,\mu}f(z)}{\mathbb{J}_p^{\lambda,\mu}f(z)}\right] \prec \frac{1+A_1z}{1+B_1z}$$

that is $f \in R_p^{\lambda,\mu}(\alpha_1; A_1, B_1)$ which implies that the assertion (3.23) of Theorem 3.12 holds.

Theorem 3.13 *Let $f(z)$ is given by equation (1.1), then*

$$|a_{p+n}| \leq \frac{\Gamma(p+n-1)\Gamma(p+n+\mu-\lambda+\nu+1)\Gamma(\nu+2)\Gamma(\mu+1)}{\Gamma(p+n+\mu)\Gamma(\mu-\lambda+\nu+1)\Gamma(p+n+\nu+1)}\left|\frac{A-B}{1+\alpha(p+n)}\right| \tag{3.27}$$

Proof. From equation (1.9), take $\phi(z) = \frac{1+Az}{1+Bz}$ we have

$$1 + \{1+\alpha(p+n)\}\left\{B_p^{\lambda+1,\mu}(p+n) - B_p^{\lambda,\mu}(p+n)\right\}a_{p+n}z^n \prec \frac{1+Az}{1+Bz} \tag{3.28}$$

An application of Lemma 2.6 to (3.28), yields

$$\left|\{1+\alpha(p+n)\}\left\{B_p^{\lambda+1,\mu}(p+n) - B_p^{\lambda,\mu}(p+n)\right\}a_{p+n}\right| \leq |A-B| \tag{3.29}$$

thus from equation (3.29), we easily arrive at (3.27) asserted by Theorem 3.13.

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