Some new results for Chebyshev matrix polynomials of first kind

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.
Department of Mathematics, College of Science and Arts, Unaizah, Qassim University, Qassim, Kingdom of Saudi Arabia.
E-mail*: drshehata2006@yahoo.com

Abstract

The main aim of the present paper is to investigate some new relations and generating matrix functions for Chebyshev matrix polynomials of the first kind. Some consequences of our main results are also discussed.

1 Introduction and preliminaries

The special matrix functions have many applications and play an important role in different branches of analysis namely infinite series, general theories of linear differential equations, statistics, operations research and functions of a complex variables, such as physics and applied mathematics, harmonic analysis, number theory, the theory of generating matrix functions has been developed into various directions etc. Many earlier (known) properties and extensions of Chebyshev matrix polynomials have been studied in [1, 5, 10, 11, 12, 13, 15, 17]. Our motivation for the important role of the Chebyshev matrix polynomials in physics and applied mathematics as applications. We organize of the paper as follows: Many results obtained for Chebyshev matrix polynomials are known but some of them are believed to be new.

Throughout this paper, for a matrix \( A \in \mathbb{C}^{N \times N} \), its spectrum is denoted by \( \sigma(A) \). The matrices \( I \) and \( O \) will denote the identity matrix and null matrix in \( \mathbb{C}^{N \times N} \), respectively. We say that a matrix \( A \) in \( \mathbb{C}^{N \times N} \) is a positive stable matrix if the real part of each of its eigenvalues is a positive. In [7], if \( \Phi(z) \) and \( \Psi(z) \) are holomorphic functions in an open set \( \Omega \) of the complex plane, and if \( A, B \) are matrices in \( \mathbb{C}^{N \times N} \) for which \( \sigma(A) \subset \Omega, \sigma(B) \subset \Omega \) and \( AB = BA \), then

\[
\Phi(A)\Psi(B) = \Psi(B)\Phi(A).
\]

From [8], for a matrix \( A \in \mathbb{C}^{N \times N} \) such that \( \sigma(A) \) does not contain 0 or a negative integer (\( \sigma(A) \cap \mathbb{Z}^- = \emptyset \) where \( \emptyset \) is an empty set), one recalls the matrix version of Pochhammer symbol as follows

\[
(A)_n = A(A+I)(A+2I)\ldots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_0 = I.
\]

(1.1)

For \( A \) is an arbitrary matrix in \( \mathbb{C}^{N \times N} \) and using (1.1), we have the following relations (Defez and Jódar [5])

\[
(A)_{n+k} = (A)_n(A+nI)_k = (A)_k(A+kI)_n,
\]

\[
(-nI)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!} I, & 0 \leq k \leq n; \\ 0, & k > n. \end{cases}
\]

(1.2)

If \( \text{Re}(\mu) \in \sigma(A) \) is not an integer and using (1.1), we have the following the relation

\[
\Gamma(I-A-nI)\Gamma^{-1}(I-A) = (-1)^n(A)_n^{-1},
\]

(1.3)

where \( \Gamma(I - A) \) is an invertible matrix.
Lemma 1.1 (Jódar and Cortés [9]) For any matrix $A$ in $\mathbb{C}^{N\times N}$, the authors exploit the following relation:

$$(1-z)^{-A} = {}_1F_0(A; -; z) = \sum_{n=0}^{\infty} \frac{1}{n!} (A)_{n} z^n; \quad |z| < 1.$$  \hspace{1cm} (1.4)

Lemma 1.2 For a matrix $A(k, n)$ in $\mathbb{C}^{N\times N}$ where $n \geq 0$, $k \geq 0$, the following relation is satisfied (see, Defez and Jódar [9])

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n - k).$$  \hspace{1cm} (1.5)

Definition 1.1 [9] For $n \geq 0$, the Jacobi matrix polynomials $P^{A,B}_n(x)$ is defined as

$$P^{(A,B)}_n(x) = \frac{(A + I)_n}{n!} {}_2F_1 \left( A + B + (n + 1)I, -nI; A + I; \frac{1-x}{2} \right),$$

$$= \frac{(-1)^n (B + I)_n}{n!} {}_2F_1 \left( A + B + (n + 1)I, -nI; B + I; \frac{1+x}{2} \right).$$  \hspace{1cm} (1.6)

where $A$ and $B$ are commutative matrices in $\mathbb{C}^{N\times N}$ satisfying the condition

$$Re(z) > -1, \quad \forall z \in \sigma(A) \quad \text{and} \quad Re(w) > -1, \quad \forall w \in \sigma(B).$$  \hspace{1cm} (1.7)

Corollary 1.1 [2] The Jacobi matrix polynomials have the following matrix recurrence relations:

$$(x-1) \left[ (A + B + nI) \frac{d}{dx} P^{(A,B)}_n(x) + (A + nI) \frac{d}{dx} P^{(A,B)}_{n-1}(x) \right]$$

$$= (A + B + nI) \left[ nP^{(A,B)}_n(x) - (A + nI)P^{(A,B)}_{n-1}(x) \right].$$  \hspace{1cm} (1.8)

Definition 1.2 [9] Let $A$ be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$0 < Re(\lambda) < 1, \quad \text{for all} \ \lambda \in \sigma(A). \hspace{1cm} (1.9)$$

The Chebyshev matrix polynomials $T_n(x, A)$ of the first kind is defined by

$$T_n(x, A) = \sum_{k=0}^{n} \frac{(-1)^k n(n + k - 1)!}{k!(n-k)!} \left( \frac{1-x}{2} \right)^k \Gamma^{-1}(A + kI) \Gamma(A), \quad n \geq 0$$

$$= \sum_{k=0}^{n} \frac{(-1)^k n(n + k - 1)!}{k!(n-k)!} \left( \frac{1-x}{2} \right)^k \Gamma^{-1}(A + kI) \Gamma(A), \quad n \geq 0$$

$$= 2F_1 \left( -nI, nI; A; \frac{1}{2}(1-x) \right) \hspace{1cm} (1.10)$$

such that $C + kI$ is an invertible matrix for all integers $k \geq 0$ and $\left| \frac{1-x}{2} \right| < 1$.

Theorem 1.1 [9] Let $A$ be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $\sigma[1]$. For $n \geq 0$, the Chebyshev matrix polynomials $T_n(x, A)$ of the first kind satisfy the pure recurrence matrix relation

$$(2n - 1)(A + nI)T_{n+1}(x, A) = \left( (1 + (2n + 1)(2n - 1)x)I - 2A \right) T_n(x, A)$$

$$+ (2n + 1)(A - nI)T_{n-1}(x, A); \quad n \geq 1. \hspace{1cm} (1.11)$$

2 Some new results for Chebyshev matrix polynomials

In order to investigate some important properties, we give the generating matrix functions and some new relations of Chebyshev matrix polynomials of the first kind in the following theorems.
Theorem 2.1 Let $A$ be matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (1.9) with $|t| < 1$ and $\left|\frac{1-x^2}{2}\right| < 1$. Then the Chebyshev matrix polynomials $T_n(x, A)$ of the first kind has the generating matrix function

$$
e^{t} {}_1F_1\left(nI; A; \frac{(1-x)t}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n(x, A). \quad (2.1)$$

Proof: Using equations (1.2), (1.4) and (1.5) in the left hand side of equation (2.2), we have

$$
e^{t} {}_1F_1\left(nI; A; \frac{(1-x)t}{2}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{n+k}}{k!n!} (nI)k[(A)k]^{-1} \left(\frac{1-x}{2}\right)^k$$

Thus, the proof is completed.

Theorem 2.2 Let $A$ and $C$ be commutative matrices in $\mathbb{C}^{N \times N}$ and $A$ satisfying the condition (1.9) with $|t| < 1$, $\left|\frac{1-x}{2}\right| < 1$ and $\left|\frac{1-x^2}{2}\right| < 1$. Then we obtain the generating matrix function for Chebyshev matrix polynomials $T_n(x, A)$ of the first kind as follows:

$$(1-t)^{-C} {}_2F_1\left(C, nI; A; \frac{(1-x)t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (C)_n T_n(x, A). \quad (2.2)$$

Proof: Using Eqs. (1.2), (1.3), (1.4) and (1.5) in the left hand side of equation (2.2), we have

$$(1-t)^{-C} {}_2F_1\left(C, nI; A; \frac{(1-x)t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (C)_n T_n(x, A).$$

Thus, the proof is completed.
Corollary 2.1 For matrices $A$ and $B$ in $\mathbb{C}^{N \times N}$ satisfying the condition (1.7), the Jacobi matrix polynomials have the following matrix recurrence relations:

\[(i) \quad (x + 1)
\frac{d}{dx} \begin{bmatrix}
(A + B + nI) P_n^{(A,B)}(x) - (B + nI) \frac{d}{dx} P_{n-1}^{(A,B)}(x)
\end{bmatrix}
= (A + B + nI)
\begin{bmatrix}
(nP_n^{(A,B)}(x) + (B + nI) P_{n-1}^{(A,B)}(x))
\end{bmatrix}.
\]

(2.3)

and

\[(ii) \quad (A + B + 2nI)(x^2 - 1) \frac{d}{dx} P_n^{(A,B)}(x) = n
\begin{bmatrix}
(B - A + (A + B + 2nI)x) P_n^{(A,B)}(x) - 2(A + nI)(B + nI) P_{n-1}^{(A,B)}(x).
\end{bmatrix}
\]

(2.4)

Proof: To prove (i): If we replace $x$ by $-x$ and interchange $A$ and $B$ with the help $P_n^{(B,A)}(-x) = (-1)^n P_n^{(A,B)}(x)$ in equation (1.6), we obtain (2.3).

To prove (ii): Let us eliminate $\frac{d}{dx} P_{n-1}^{(A,B)}(x)$ from (1.9) and (2.3), we obtain the result (2.4), we complete the proof.

Theorem 2.3 For a matrix $A$ in $\mathbb{C}^{N \times N}$ satisfying the condition (1.9), Then the Chebyshev matrix polynomials defined in (1.10) satisfy the matrix differential recurrence relations

\[(i) \quad (x - 1(n-1)DT_n(x,A) + nDT_{n-1}(x,A)) = n(n-1)(T_n(x,A) - T_{n-1}(x,A)); n \geq 1,
\]

(2.5)

\[(ii) \quad (x + 1)(n-1)(A + (n-1)I)DT_n(x,A) + n(A - nI)DT_{n-1}(x,A)
\]

\[= n(n-1)(A + (n-1)I)T_n(x,A) - (A - nI)T_{n-1}(x,A)); n \geq 1,
\]

(2.6)

and

\[(iii) \quad (2n-1)(x^2 - 1)DT_n(x,A) = n(1 + (2n-1)x)I - 2A)T_n(x,A) + 2n(A - nI)T_{n-1}(x,A); n \geq 1.
\]

(2.7)

Theorem 2.4 Let $A$ be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (1.9). Then the Chebyshev matrix polynomials of first kind satisfy

\[(2n + 1)(x^2 - 1)DT_n(x,A) = n(1 - (2n + 1)x)I - 2A)T_n(x,A) + 2n(A + nI)T_{n+1}(x,A).
\]

(2.8)

Proof: If we eliminate $T_{n-1}(x,A)$ from (1.11) and (2.7) which leads to the result (2.8).

Remark 2.1 For the scalar case $N = 1$, taking $A = \frac{1}{2}$, $T_n(x, \frac{1}{2})$, gives the classical scalar Chebyshev polynomials of the first kind in [3, 4, 7, 10].

Remark 2.2 Taking $A = \frac{1}{2}$ in Theorems (2.3) and (2.4), we obtain some recurrence relations for the scalar Chebyshev polynomials of the first kind given in [3, 4, 7, 10].

3 Conclusion

In the present paper, we have investigated some new relations and generating matrix functions for Chebyshev matrix polynomials. Also, we have investigated many important properties and consequences of our main results. Since Chebyshev matrix polynomials are associated with a wide range of problems in diverse fields of mathematical physics, biology, engineering and applied sciences. The results of this Chebyshev matrix polynomials can be studied in further
research. The results thus derived in this paper are general in character and likely to find certain applications in the theory of special matrix functions.

ACKNOWLEDGMENT

The author would like to thank the anonymous referees for their valuable comments and suggestions that are given for improving the presentation of the present paper. Particularly, the author would like to thank greatly the anonymous referees who evoked questions that allowed to make clear the deep insights of this paper and its impact on future work within this line of research.

References


Ayman Shehata is currently Assistant Professor, Department of Mathematics, College of Science and Arts, Unaizah, Qassim University, Qassim, Kingdom of Saudi Arabia. He completed B.Sc. (Mathematics) in 2000, M.Sc. (Applied Mathematics) in 2006 and Ph.D. degree (Pure Mathematics) in 2009 from The Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt. He has more than 18 years of academic and research experience. His research field is in mathematical, complex analysis, mathematical inequalities, especially of special functions, multiple hypergeometric series, basic hypergeometric series, generalized functions, integral transforms, fractional calculus, functions of matrix arguments and orthogonal matrix polynomials and their applications. He has published more than 55 research papers in the national and international mathematics journals of repute. He has also served as an Editorial Board and reviewer for a number of several national/international journals of pure and applied mathematics. He is a member of the Egyptian Mathematical Society.