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Legendre Polynomials in Fractional Extensions

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Abstract

This paper refers to fractional order generalization of classical Legendre polynomials. Rodrigues type representation formula of fractional order considered. Fractional Legendre functions are defined by means of the Riemann-Liouville fractional calculus operator.

1 Introduction and Preliminaries

The fractional calculus becomes one of the intensively developing areas of mathematical analysis. It's fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. On behalf of the nature of their definition, the fractional derivatives provide of various materials and processes. For example, the half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems then the classical method [2]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equation [14] and recently of the temperature field problem in oil strata [1].

Generalizing Rodrigues formula for the classical Legendre polynomial by means of the Riemann-Liouville fractional differentiation operator. We define the so-called fractional Legendre function.

Definition 1. Let $f(t)$ be piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $(0,\infty)$.

(i) The Riemann-Liouville fractional integral of $f(t)$ of order α is defined

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(t)\, dt. \tag{1}$$

The operators are defined by Samko et al. [11] for $\alpha > 0$

$$\left[I_{0+}^{\alpha} f(t)\right] = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \tag{2}$$

$$\left[I_{0-}^{\alpha} f(t)\right] = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\, dt. \tag{3}$$

(ii) The Riemann-Liouville fractional derivative of $f(t)$ of order μ is defined by

$$D^{\mu} f(t) = D^m \left[J^{m-\mu} f(t)\right], \tag{4}$$

again

$$\left[D_{0+}^{\alpha} f(t)\right] = \left(\frac{d}{dx}\right)^{\alpha+1} \left[I_{0+}^{1-\alpha} f\right](x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right)^{\alpha+1} \int_0^x \frac{f(t)}{(x-t)^{\alpha}}\, dt, \tag{5}$$

$$\left[D_{-}^{\alpha} f(t)\right] = \left(-\frac{d}{dx}\right)^{\alpha+1} \left[I_{-}^{1-\alpha} f\right](x) = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dx}\right)^{\alpha+1} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha}}\, dt. \tag{6}$$

For later considerations we need just to mention that if $\mu \geq 0$, $t > 0$ and $\alpha > -1$, then the fractional derivative of the

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power function t^{α} is given by Boyadjiev and Scherer [1] as

$$D^{\mu}\, t^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\mu)}\, t^{\alpha-\mu}. \tag{7}$$

It can be shown [9, 7] that formula (7) hold for negative values of μ as well, and in this case D^{μ} is to be considered as J^{μ}.

Also, we essentially use the Leibniz rule for fractional differentiation [9], that for a continuous on $[0, t]$ function $f(x)$ and continuously differentiable on the same interval function $\varphi(x)$ takes the form

$$D^{\mu}\,[\varphi(t)\ f(t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} \varphi^{k}(t)\ D^{\mu-k} f(t)\,. \tag{8}$$

Consider also the Gauss hypergeometric function for $|x| < 1$ (see, e.g. [10])

$$_{2}F_{1}(a, b, c;\, x) \ = \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k}{(c)_k}\, \frac{x^k}{k!}, \tag{9}$$

where $(x)_n$ is the Pochhammer symbol.

$$(x)_n = \frac{\Gamma\ (x+n)}{\Gamma(x)} = x\ (x+1)\ \ldots\ (x+n-1)\,. \tag{10}$$

2 Fractional Legendre Function

The classical Legendre polynomials are usually defined by means of Rodrigues formula

$$P_n(x) = \frac{1}{2^n\, n!}\, D^n\ \left(x^2-1\right)^n, \tag{11}$$

using Leibniz rule for the n^{th} derivative of a product is

$$D^{\mu}\,[u, v] \ = \sum_{k=0}^{\infty} C_{n,k} \left(D^k\, u\right) \left(D^{n-k}\, v\right), \tag{12}$$

in which $D = \frac{d}{dx}$ and u and v are to be function at x. The validity of (12) is easily show by induction. Since, by Rodrigues formula

$$P_n(x) = \frac{1}{2^n\, n!}\, D^n\, [(x-1)^n\ (x+1)^n]\,. \tag{13}$$

Definition 2. The fractional Legendre function is defined by Nishimoto [8].

$$_{c}L_{\,v}(t) \ = \ \frac{1}{2^v\, \Gamma(v+1)} D^v\ \left[(t-1)^v\ (t+1)^v\right]. \tag{14}$$

Taking into account that the binomial coefficients with real argument is defined as

$$\binom{\alpha}{\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\ \Gamma(1+\alpha-\beta)}, \tag{15}$$

the generalized binomial theorem follows as:

$$D^{v-k}\,(1-t)^v = \sum_{k=0}^{\infty} \binom{v}{r} (-1)^{v-r}\ D^{v-k}\, t^{v-r}. \tag{16}$$

3 Main Results

Theorem 1 *For the fractional Legendre function the following representation holds*

$$ {}_cL_v(t) = (-1)^{v-2r}\ (2)^{-v} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v}{k} (t+1)^{v+k} (t-1)^k . \tag{17}$$

Proof Using (12) and (14), we have the following:

$$\begin{aligned} {}_cL_v(t) &= \frac{1}{2^v\,\Gamma(v+1)} D^v\ [(t-1)^v (t+1)^v] \\ &= \frac{1}{2^v\,\Gamma(v+1)} \sum_{k=0}^{\infty} \binom{v}{k}\ D^k(t+1)^v\, D^{v-k}(1-t)^v \\ &= \frac{1}{2^v\,\Gamma(v+1)} \sum_{k=0}^{\infty} \binom{v}{k}\ \frac{\Gamma(1+v)}{\Gamma(1+v-k)} (t+1)^{v-k}\, D^{v-k}(1-t)^v \\ &= \frac{1}{2^v\,\Gamma(v+1)} \sum_{k=0}^{\infty} \binom{v}{k}\ \frac{\Gamma(1+v)}{\Gamma(1+v-k)} (t+1)^{v-k} \sum_{r=0}^{\infty} \binom{v}{r}\ (-1)^{v-r} D^{v-k}\, t^{v-r} \\ &= \frac{(-1)^v}{2^v} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v}{k} (t+1)^{v-k}\, \frac{(t-1)^k}{(-1)^{2r}}. \end{aligned} \tag{18}$$

Corollary 1.1 *Let the condition of Theorem 1 be valid and replace t by $-t$, then we arrive at the following result:*

$$ {}_cL_v\ (-t) = (-1)^{v-2r+k}\ (2)^{-v} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v}{k}\ (1-t)^{v-k} (1+t)^k . \tag{19}$$

Corollary 1.2 *Let the condition of Theorem 1 be valid and $t=-1$, then we have the following result:*

$$ {}_cL_v\ (-1) = (-1)^{v-2r}\ (2)^{-v} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v}{k} (-1+1)^{v+k} (-1-1)^k $$

That is,

$$ {}_cL_v\ (-1) = 0. \tag{20}$$

Corollary 1.3 *The differentiation of (17) is given as*

$$\begin{aligned} \frac{d}{dt}\ [{}_cL_v(t)] &= \frac{d}{dt}\left[(-1)^{v-2r}\, (2)^{-v} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v}{k}\ (t+1)^{v+k} (t-1)^k\right] \\ &= [v\ (1-t) - 2kt]\ {}_cL_v(t). \end{aligned} \tag{21}$$

Theorem 2 *The fractional Legendre function can be represented as*

$$ {}_cL_v(t) = (-1)^{v+k} \sum_{k=0}^{\infty} \binom{v}{k} \binom{v+k}{v} \left(\frac{t+1}{2}\right)^k . \tag{22}$$

Proof : In the fractional Legendre function

$$ {}_cL_v(t) = (-1)^v\, {}_2F_1\left[-v,\ v+1,\ 1,\ \frac{1+t}{2}\right] = (-1)^v \sum_{k=0}^{\infty} \frac{(-v)_k\ (v+1)_k}{(1)_k\, k!} \left(\frac{t+1}{2}\right)^k ,$$

if we use $(-x)_n = (-1)^n (x-n+1)_n$, then we arrive at

$$\begin{aligned}
{}_cL_v &= (-1)^v \sum_{k=0}^{\infty} \frac{(-1)^k\ (v-k+1)_k\,(v+1)_k}{(1)_k\ k!}\ \left(\frac{t+1}{2}\right)^k \\
&= (-1)^v \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(v-k+1+k)\ \Gamma(1+v+k)\ \Gamma(1)}{\Gamma(1+v-k)\ \Gamma(1+v)\ \Gamma(1+k)\ \Gamma(1+k)\ k!} \left(\frac{t+1}{2}\right)^k \qquad (23) \\
&= (-1)^v \sum_{k=0}^{\infty} \binom{v}{k} \binom{v+k}{v} \left(\frac{t+1}{2}\right)^k .
\end{aligned}$$

4 Concluding Remarks

In the present paper we have defined the fractional order generalization of classical Legendre polynomials. We obtained the Rodrigues type representation formula of fractional order. It is also shown that the main results are general enough to be specialized yield many known or new results.

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